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COMPUTATION OF EFFECTIVE ELASTIC PROPERTIES OF MAGNETOELASTIC COMPOSITES USING HOMOGENIZATION METHODS

Abstract. The main aim of this paper is to provide methods of computing effective elastic properties of magnetoelastic composites in macroscale in two cases: when filling particles are distributed randomly and when they form chain structures due to the precense of external magnetic field during the fabrication process. Presented approach uses standard homogenization methods based on Eshelby's eigenstrain principle and their extension known as self-consistent method

Keywords: magnetoelastic composites, homogenization, eigenstrain, Eshelby's tensor, self-consistent method

WYZNACZANIE EFEKTYWNYCH WŁAŚCIWOŚCI SPRĘŻYSTYCH KOMPOZYTÓW MAGNETOELASTYCZNYCH Z WYKORZYSTANIEM METOD HOMOGENIZACJI

Streszczenie: W artykule przedstawione zostały metody obliczania efektywnych właściwości sprężystych kompozytów magnetoelastycznych w skali makro, w dwóch przypadkach: gdy cząsteczki są rozmieszczone losowo oraz gdy tworzą one struktury łańcuchowe pod wpływem działania zewnętrznego pola magnetycznego w trakcie procesu wytwarzania. Zaprezentowane podejście wykorzystuje standardowe metody homogenizacji oparte na zasadzie naprężeń własnych Eshelby'ego oraz ich rozszerzenia znanego powszechnie jako metoda samouzgodnionego pola.

Słowa kluczowe: kompozyty magnetoelastyczne, homogenizacja, naprężenie własne, tensor Eshelby'ego, metoda samouzgodnionego pola

Introduction

The principle of this paper is to present methods of homogenization enabling calculation of effective elastic properties of magnetoelastic composites assuming finite strain, in two states: with no external magnetic field, when ferromagnetic filling particles are distributed randomly in matrix and in presence of such field, when

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particles form parallel chains. Such chains may be considered as fiber reinforcement of composite and consequently effective elastic properties can be computed using homogenization methods for composites consisting of isotropic matrix and cylindrical inclusions.
Firstly, magnetoelastic composites will be introduced. Then some basics of homogenization methods will be presented. Subsequently, methodology of computing effective magnetic properties of magnetoelastic composites in two analyzed states will be shown.

1. Magnetoelastic composites

Magnetoelastic composites belong to the wider group of SMART materials – magnetorheological materials which exhibit dependence of rheological properties on magnetic field. They consist of isotropic elastic matrix with embedded small ferromagnetic particles. Commonly polyurethane, natural rubber or silicone rubber states as matrix materials, whereas carbonyl iron or terfenol-D particles are used as ferromagnetic filling [7].
When no external magnetic field is applied, particles are distributed randomly in matrix (fig. 1 left). However, presence of such field during fabrication of composites causes particles to form parallel chains longitudinal to the lines of external field (fig. 1 right).

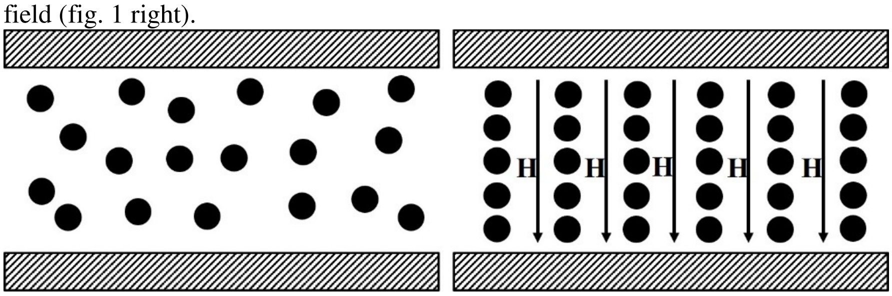

Figure 1: Visual model of filling particles distribution in magnetoelastc compostite with no external magnetic field (left) and with applied external magnetic field (right)

After matrix material curing, particles are locked and consequently chain structure is sustained after removal of magnetic field. Presence of such chains increases macroscopic stiffness of material. Moreover, further application of magnetic field causes effective magnetostriction of material due to the interactions between magnetized particles [7].
Modelling of magnetorheological materials is rather a difficult task and requires analyzing mutual interactions between electromagnetic and elastic fields in material [4]. Assuming quasi-balanced state of magnetic saturation of ferromagnetic particles, when material magnetization is constant, elastic properties of material may be treated as constant. Consequently, effective elastic properties can be find via standard homogenization methods basing on so-called Eshelby tensor. However it still requires finding properties of chains in mesoscale. Estimation of elements of stiffness tensor for assumed chains would exceed the volume of this paper and will be omitted. Main

focus will be finding effective stiffness tensors in macroscale, assuming that stiffness tensor for chains is known.

2. Preliminaries of homogenization methods

Main aim of mechanical homogenization is to find relation between average strain and stress fields of material in macroscale given by (1), knowing constitutive equations in micro or mesoscale.

$$<\boldsymbol{\varepsilon}>=\frac{1}{|V|}\int_V \varepsilon(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \qquad <\boldsymbol{\sigma}>=\frac{1}{|V|}\int_V \sigma(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \tag{1}$$

where V denotes a volume element.

Homogenization is widely used to estimate effective mechanical properties of heterogenous materials, especially composites which consist of phases exhibiting different mechanical properties [5]. Mechanical homogenization methods used in engineering applications may be classified into two general groups: first basing on Eshelby eigenstrain principle, where inhomogeneities may be replaced by stress and strain fields which causes equivalent mechanical effects and the second one, called energetic methods using variational calculus to minimize certain functionals, mostly energy or free energy functions. Eigenstrain methods are very effective in linear elasticity problems, whereas in nonlinear cases energetic methods give better results. Consider a composite material consisting of two clearly elastic phases – matrix denoted as M and inclusions – Ω and assume that both, matrix and inclusions are isotropic. Let $\boldsymbol{C}$ and $\boldsymbol{C}^{\boldsymbol{\Omega}}$ be stiffness tensors of matrix and inclusions respectively. The task is to find effective stiffness tensor $\overline{\boldsymbol{C}}$ enforcing relation (2).

$$<\boldsymbol{\sigma}>=\overline{\boldsymbol{C}}:<\boldsymbol{\varepsilon}> \tag{2}$$

Fundamental concept is to divide composite into domains including exactly one inhomogeneity. Such domains are known in literature as *Representative volume elements (RVE).* It is noticeable that there are no size limitations for RVE, the only restriction is that it must include exactly one inhomogeneity. Denote RVE as V, inclusion as Ω and matrix in RVE as $M = V\backslash\Omega$. Considering that RVE is subjected to a traction boundary condition $t = \boldsymbol{p}^{\boldsymbol{T}} \cdot \boldsymbol{n}$ on the boundary ∂V, stress and strain fields in RVE may be decomposed into average stress and strain and perturbation of the field caused by the presence of inhomogeneity:

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{\boldsymbol{0}} + \boldsymbol{\varepsilon}^{\boldsymbol{d}} \qquad \boldsymbol{\sigma} = \boldsymbol{\sigma}^{\boldsymbol{0}} + \boldsymbol{\sigma}^{\boldsymbol{d}} \tag{3}$$

Eshelby's approach was to superpose the actual strain field $\boldsymbol{\varepsilon}$ with so-called eigenstrain $\boldsymbol{\varepsilon}^*$ for which $\mathrm{supp}(\varepsilon^*) = \Omega$ such that for all points $\boldsymbol{x} \in \Omega$ equation (4) holds.

$$\boldsymbol{C}^{\boldsymbol{\Omega}}:\left(\boldsymbol{\varepsilon}^{\boldsymbol{0}}(\boldsymbol{x}) + \boldsymbol{\varepsilon}^{\boldsymbol{d}}(\boldsymbol{x})\right) = \boldsymbol{C}:\left(\boldsymbol{\varepsilon}^{\boldsymbol{0}}(\boldsymbol{x}) + \boldsymbol{\varepsilon}^{\boldsymbol{d}}(\boldsymbol{x}) - \boldsymbol{\varepsilon}^*(\boldsymbol{x})\right) \tag{4}$$

Eshelby shown that if an inhomogeneity is an ellipsoid then $\boldsymbol{\varepsilon}^* = const$ for all $\boldsymbol{x} \in \Omega$ and it is may be related to $\boldsymbol{\varepsilon}^{\boldsymbol{d}}$ by equation (5) [1].

$$\boldsymbol{\varepsilon}^{\boldsymbol{d}} = \boldsymbol{S}:\boldsymbol{\varepsilon}^* \tag{5}$$

where $\boldsymbol{S}$ is fourth order tensor, known in literature as Eshelby tensor.
Assuming that inhomogeneity Ω is placed in infinite linear-elastic medium, it may be shown that eq. (5) takes form (6).

$$\varepsilon_{ij}^{d}(\boldsymbol{x}) = \varepsilon_{mn}^{*} \int_{\Omega} -\frac{1}{2} C_{klmn} \left(G_{ik,jl}(\boldsymbol{x}-\boldsymbol{y}) + G_{jk,il}(\boldsymbol{x}-\boldsymbol{y}) \right) \mathrm{d}\mathbf{y} \tag{6}$$

where $G_{ij}(\boldsymbol{x}-\boldsymbol{y})$ are components of Green tensor derived for isotropic, linear elastostatics equilibrium equation with boundary condition $u_{ij}(\boldsymbol{x}) \to 0$ when $\boldsymbol{x} \to \infty$.
Integral in (6) is the S_{ijmn} component of Eshelby tensor $\boldsymbol{S}$.
Green tensor components $G_{ij}(\boldsymbol{x}-\boldsymbol{y})$ are given by equation (7).

$$G_{ik}(\boldsymbol{x}-\boldsymbol{y}) = \frac{1}{4\pi\mu} \left(\delta_{ik} \frac{1}{||\boldsymbol{x}-\boldsymbol{y}||} - \frac{\lambda+\mu}{\lambda+2\mu} \frac{(x_i-y_i)(x_k-y_k)}{||\boldsymbol{x}-\boldsymbol{y}||^3} \right) \tag{7}$$

where λ and μ are Lamé constants of matrix.
Introducing potential functions $\phi(\boldsymbol{x})$ and $\psi(\boldsymbol{x})$ defined as (8) and (9) one may show that S_{ijkl} takes form (10) [1].

$$\phi(\boldsymbol{x}) = \int_{\Omega} \frac{1}{||\boldsymbol{x}-\boldsymbol{y}||} \mathrm{d}\boldsymbol{y} \tag{8}$$

$$\psi(\boldsymbol{x}) = \int_{\Omega} ||\boldsymbol{x}-\boldsymbol{y}|| \mathrm{d}\boldsymbol{y} \tag{9}$$

$$S_{ijkl}(\boldsymbol{x}) = -\frac{1}{8\pi\mu} C_{klmn} \Big(\delta_{ik}\phi_{,jl}(\boldsymbol{x}) + \delta_{jk}\phi_{,il}(\boldsymbol{x}) - 2\frac{\lambda+\mu}{\lambda+2\mu} \psi_{,ikjl}(\boldsymbol{x}) \Big) \tag{10}$$

If Ω is an ellipsoid, then function $\phi(\boldsymbol{x})$ satisfies Laplace equation $\Delta\phi(\boldsymbol{x}) = 0$ for $\boldsymbol{x} \in \Omega$, so it is so-called newtonian potential [2]. Moreover function $\psi(\boldsymbol{x})$ is related to $\phi(\boldsymbol{x})$ by equation (11) what may be checked by direct calculus.

$$\Delta\psi(\boldsymbol{x}) = 2\phi(\boldsymbol{x}), \qquad \boldsymbol{x} \in \Omega \tag{11}$$

Hence $\psi(\boldsymbol{x})$ is biharmonic function inside ellipsoid Ω. It may be derived that $\phi(\boldsymbol{x})$ is given by equation (12) for $\boldsymbol{x}$ inside ellipsoid Ω.

$$\phi(\boldsymbol{x}) = 2\pi abc \int_0^{+\infty} \left(1 - \frac{x_1^2}{a^2+v} - \frac{x_2^2}{b^2+v} - \frac{x_3^2}{c^2+v} \right) \frac{\mathrm{d}v}{\sqrt{f(v)}} \qquad f(v) = (a^2+v)(b^2+v)(c^2+v) \tag{12}$$

where a, b, c are axes of the ellipsoid and coordinates of x_1, x_2, x_3 are parallel to a, b and c respectively.
From equations (12) and (11) it follows that $\phi(\boldsymbol{x})$ and $\psi(\boldsymbol{x})$ are both polynomials of variable $\boldsymbol{x} \in \Omega$ of degrees 2 and 4 respectively. Hence $\boldsymbol{S}$ is constant for all $\boldsymbol{x} \in \Omega$. Moreover elements S_{ijkl} may be expressed by elliptic integrals of first and second find and consequently easily computed using numerical methods [5].
In specific case, when Ω is a sphere with a radius r elements of $\boldsymbol{S}$ take form (13).

$$S_{ijkl} = \frac{3\lambda-2\mu}{15(\lambda+2\mu)} \delta_{ij}\delta_{kl} + \frac{3\lambda+8\mu}{15(\lambda+2\mu)} (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{13}$$

Whence Eshelby tensor $\boldsymbol{S}$ for a spherical inhomogeneity exhibits symmetries characteristic for isometric second order tensors. When inhomogeneity has ellipsoidal shape, Eshelby tensor $\boldsymbol{S}$ is not isotropic but exhibits some symmetries as well.
Considering the limit case $a \to \infty$, Eshelby tensor for cylindrical inhomogeneities may be derived. In such case, potentials $\phi(\boldsymbol{x})$ and $\psi(\boldsymbol{x})$ must be independent of x_1 and isotropic in the plane x_2x_3 perpendicular to x_1. Hence, in that case, $\boldsymbol{S}$ is

transversely isotropic fourth order tensor. For isotropic material of matrix, Eshelby tensor elements S_{ijkl} are given by equation (14) [3].

$$S_{ijkl} = \frac{\lambda+\mu}{4(\lambda+2\mu)}\Theta_{ij}\Theta_{kl} + \frac{\lambda+3\mu}{8(\lambda+2\mu)}\left(\Theta_{ik}\Theta_{jl} + \Theta_{il}\Theta_{jk} - \Theta_{ij}\Theta_{kl}\right) + \frac{\lambda}{2(\lambda+2\mu)}\Theta_{ij}\delta_{k1}\delta_{l1} + \frac{1}{4}(\Theta_{ik}\delta_{l1}\delta_{j1} + \Theta_{il}\delta_{k1}\delta_{j1} + \Theta_{jk}\delta_{l1}\delta_{i1} + \Theta_{jl}\delta_{l1}\delta_{i1}) \tag{14}$$

where $\Theta_{ij} = \delta_{ij} - \delta_{i1}\delta_{j1}$

In case of chains of ferromagnetic particles treated as fibres it is expected that estimated stiffness tensor $\widetilde{\boldsymbol{C}}$ of a chain is not isotropic but transversely isotropic and has similar form to (14) [7].

3. Homogenization for randomly distributed inclusions

When no external magnetic field is applied, particles are distributed randomly in matrix. In that case, calculating effective elastic properties (Young modulus and Poisson ratio) may be performed via homogenization using Hill or Eshelby tensor method (eigenstrain method). Mostly, volume fraction of ferromagnetic inclusion in magneto-elastic composites exceeds 50% (reaching even 80%). Hence, the interaction among inhomogeneities cannot be ignored. Analysing interactions between stress and strain fields connected with each inclusion separately is of course very laborious task. In that case some averaging methods are used instead. Two most commonly used will be briefly described.

In first one, proposed by Hill, matrix stiffness tensor $\boldsymbol{C}$ is substituted by effective stiffness tensor of composite $\overline{\boldsymbol{C}}$ and similarly Eshelby tensor $\boldsymbol{S}$ for the inclusion is substituted by effective tensor $\overline{\boldsymbol{S}}$. Tensor $\overline{\boldsymbol{C}}$ is computed via standard homogenization from (1) and then $\overline{\boldsymbol{S}}$ is calculated from (2), assuming $\lambda = \bar{\lambda}$ and $\mu = \bar{\mu}$. This method is commonly known as *self-consistent method.* Finding effective elastic properties of the composite material via this method requires computing Eshelby tensor and effective stiffness tensor $\overline{\boldsymbol{C}}$ twice [5].

Second method uses Mori-Tanaka lemma stating that if an ellipsoidal particle Ω_0 is placed in ellipsoidal domain Ω such that Ω and Ω_0 are similar and coaxial, then the average disturbance strain field caused by the disparity of elastic properties of matrix and inclusion is zero [5]. In this method the interaction between inclusions in composite is taken into account by introducing tensor $\boldsymbol{B}$ which relates strain field in matrix with the average strain field in RVE [6].

$$< \varepsilon >_M = \boldsymbol{B} : < \boldsymbol{\varepsilon} >$$
$$\boldsymbol{B} = \left[\left(1 - \frac{|\Omega|}{|V|}\right)\mathbf{1}^{(4s)} + \frac{|\Omega|}{|V|}\boldsymbol{\mathcal{A}}\right]^{-1} \tag{15}$$
$$\boldsymbol{\mathcal{A}} = \left[\mathbf{1}^{(4s)} + \boldsymbol{P} : (\boldsymbol{C}^{\boldsymbol{\Omega}} - \boldsymbol{C})\right]^{-\mathbf{1}}$$

where $\boldsymbol{C}$ and $\boldsymbol{C}^{\boldsymbol{\Omega}}$ are stiffness tensors of matrix and inclusion respectively, $\boldsymbol{P}$ is so-called Hills polarization tensor equal to $\boldsymbol{P} = \boldsymbol{S} : \boldsymbol{C}^{-\mathbf{1}}$ and $\boldsymbol{S}$ is Eshelby tensor.

In both cases: with no external magnetic field applied, when inclusions are randomly distributed in matrix and after applying magnetic field, when inclusions form parallel chains, self-consistent method can be used to compute effective elastic properties of the composite. For randomly distributed inclusions, inverse effective stiffness tensor $\overline{\boldsymbol{D}} = \overline{\boldsymbol{C}}^{-1}$ may be computed from equation (16).

$$\overline{\boldsymbol{D}} = \left[\mathbf{1}^{(4s)} + \frac{|\Omega|}{|V|}((\boldsymbol{C} - \boldsymbol{C}^{\Omega})^{-1} : \boldsymbol{C} - \boldsymbol{S})\right] : \boldsymbol{D} \tag{16}$$

Assuming spherical shape of inclusions, Eshelby tensor $\boldsymbol{S}$ takes form (13) and is isotropic fourth order tensor as well as $\boldsymbol{C}$, $\boldsymbol{D}$ and $\boldsymbol{C}^{\Omega}$. Each fourth order isotropic tensor $\boldsymbol{A}$ may be written in form (17) [5].

$$\boldsymbol{A} = a\mathbf{1}^{(2)} \otimes \mathbf{1}^{(2)} + 2b\mathbf{1}^{(4s)} \tag{17}$$

In order to simplify algebraic operations on such tensors it is convenient to represent them as combinations of tensors $\boldsymbol{E}^{1}$ and $\boldsymbol{E}^{2}$ given by equations (18) and (19).

$$\boldsymbol{E}^{1} = \frac{1}{3}\mathbf{1}^{(2)} \otimes \mathbf{1}^{(2)} \tag{18}$$

$$\boldsymbol{E}^{2} = -\frac{1}{3}\mathbf{1}^{(2)} \otimes \mathbf{1}^{(2)} + \mathbf{1}^{(4s)} \tag{19}$$

Tensors $\boldsymbol{E}^{1}$ and $\boldsymbol{E}^{2}$ have following useful properties:

$$\boldsymbol{E}^{1} : \boldsymbol{E}^{1} = \boldsymbol{E}^{1} \tag{20}$$

$$\boldsymbol{E}^{2} : \boldsymbol{E}^{2} = \boldsymbol{E}^{2} \tag{21}$$

$$\boldsymbol{E}^{1} : \boldsymbol{E}^{2} = \boldsymbol{E}^{2} : \boldsymbol{E}^{1} = \mathbf{0} \tag{22}$$

Using above expressions, double contraction of two isotropic fourth order tensors $\boldsymbol{A}_{1}$ and $\boldsymbol{A}_{2}$ represented as combination $\boldsymbol{A}_{1} = \alpha_1 \boldsymbol{E}^{1} + \beta_1 \boldsymbol{E}^{2}$ and $\boldsymbol{A}_{2} = \alpha_2 \boldsymbol{E}^{1} + \beta_2 \boldsymbol{E}^{2}$ may be expressed by (23). Moreover, inverse tensor $\boldsymbol{A}^{-1}$ takes form (24).

$$\boldsymbol{A}_{1} : \boldsymbol{A}_{2} = \alpha_1\alpha_2 \boldsymbol{E}^{1} + \beta_1\beta_2 \boldsymbol{E}^{2} \tag{23}$$

$$\boldsymbol{A}^{-1} = \frac{1}{\alpha_1}\boldsymbol{E}^{1} + \frac{1}{\alpha_2}\boldsymbol{E}^{2} \tag{24}$$

One may show that tensor $\overline{\boldsymbol{D}}$ is given by equation (25).

$$\begin{gathered}\overline{\boldsymbol{D}} = L_1 \boldsymbol{E}^{1} + L_2 \boldsymbol{E}^{2} \\ L_1 = \frac{1}{\lambda + 2\mu}\left(1 + \frac{|\Omega|}{|V|}\frac{1}{\frac{\lambda + 2\mu}{\lambda + 2\mu - 3\lambda^{\Omega} - 2\mu^{\Omega}} - \frac{3\lambda + 2\mu}{3(\lambda + 2\mu)}}\right) \\ L_2 = \frac{1}{2\mu}\left(1 + \frac{|\Omega|}{|V|}\frac{1}{\frac{\mu}{\mu - \mu^{\Omega}} - \frac{2(3\lambda + 8\mu)}{15(\lambda + 2\mu)}}\right)\end{gathered} \tag{25}$$

where $\lambda, \mu, \lambda^{\Omega}, \mu^{\Omega}$ are Lamé constants of matrix and inclusion respectively.
Using the fact that $\boldsymbol{D}$ tensor for isotropic material may be written in form (26), effective Lamé constants takes form (27).

$$\boldsymbol{D} = \frac{1}{\lambda + 2\mu}\boldsymbol{E}^{1} + \frac{1}{2\mu}\boldsymbol{E}^{2} \tag{26}$$

$$\bar{\lambda} = \frac{1}{L_1} - \frac{1}{L_2} \qquad \bar{\mu} = \frac{1}{4L_2} \tag{27}$$

Above expressions are true for dilute composites with relatively low volume fraction of inclusions. Lamé constants for non-dilute case with interactions between inhomogeneities taken into account using self-consistent method may be computed from equations (25) and (27) by substituting λ and μ by $\bar{\lambda}$ and $\overline{\mu}$.

4. Homogenization for inclusions ordered in chains

In presence of external magnetic field ferromagnetic particles in magnetoelastic composites form chains which axes are parallel to the magnetic field lines. Such chains may be treated as fiber reinforcement of the composite and consequently effective elastic properties in the macroscale can be derived via Eshelby eigenstrain homogenization method for cylindrical inhomogeneities with circular cross-section. Such approach requires finding elastic properties of fibers equivalent to properties of chains. As it was mentioned in section 2, it should be expected that chains treated as fibers exhibit symmetry properties characteristic for transversely isotropic medium. Stiffness tensor $\boldsymbol{C}$ for transversely isotropic materials is characterized by 5 independent parameters and its elements C_{ijkl} may be written in form (28). Independent parameters of $\boldsymbol{C}$ in (28) are indexed as in Voigt notation for symmetric fourth order tensor.

$$\begin{aligned} C_{ijkl} &= C_{11}\Delta_1 + C_{22}(\Delta_2+\Delta_3) + C_{12}(\Theta_{ij}\delta_{k1}\delta_{l1} + \Theta_{kl}\delta_{i1}\delta_{j1}) + \\ &+ C_{23}\delta_{ij}\delta_{kl}(\delta_{i2}\delta_{k3} + \delta_{i3}\delta_{k2}) + \frac{C_{44}}{2}(\eta_{12}+\eta_{13}) + \frac{C_{22}-C_{23}}{4}\eta_{23} \end{aligned} \tag{28}$$

where $\Delta_q = \delta_{iq}\delta_{jq}\delta_{kq}\delta_{lq}$ and $\eta_{qr} = (\delta_{iq}\delta_{jr} + \delta_{ir}\delta_{jq})(\delta_{kq}\delta_{lr} + \delta_{kr}\delta_{lq})$

Similarly to the case, when filling particles were distributed randomly in matrix, to perform algebraic operations on tensor $\boldsymbol{C}$ it is convenient to represent it as a linear combination of so-called Hill basis tensors $\boldsymbol{H^s}, s = 1, \dots, 6$, which elements are given by equation (29) [3].

$$\begin{aligned} &H^1_{ijkl} = \frac{1}{2}\Theta_{ij}\Theta_{kl}, \qquad H^2_{ijkl} = \Theta_{ij}\delta_{k1}\delta_{l1}, \qquad H^3_{ijkl} = \Theta_{kl}\delta_{i1}\delta_{j1} \\ &H^4_{ijkl} = \delta_{i1}\delta_{j1}\delta_{k1}\delta_{l1}, \qquad H^5_{ijkl} = \frac{1}{2}(\Theta_{ik}\Theta_{jl} + \Theta_{il}\Theta_{jk} - \Theta_{ij}\Theta_{kl}) \\ &H^6_{ijkl} = \frac{1}{2}(\Theta_{ik}\delta_{l1}\delta_{k1} + \Theta_{il}\delta_{k1}\delta_{j1} + \Theta_{jk}\delta_{l1}\delta_{i1} + \Theta_{jl}\delta_{k1}\delta_{i1}) \end{aligned} \tag{29}$$

Hill basis tensors $\boldsymbol{H^k}$ satisfy double contraction relations summarized in (30) [3].

$$\begin{aligned} &\boldsymbol{H^1}:\boldsymbol{H^s} = \boldsymbol{2H^4}\delta_{s2} + \boldsymbol{H^1}\delta_{s3} \qquad \boldsymbol{H^2}:\boldsymbol{H^s} = 2\boldsymbol{H^3}\delta_{s1} + \boldsymbol{H^2}\delta_{s4} \\ &\boldsymbol{H^3}:\boldsymbol{H^s} = \boldsymbol{H^2}\delta_{s2} + \boldsymbol{H^3}\delta_{s3} \qquad \boldsymbol{H^4}:\boldsymbol{H^s} = \boldsymbol{H^1}\delta_{s1} + \boldsymbol{H^4}\delta_{s4} \\ &\boldsymbol{H^5}:\boldsymbol{H^s} = \boldsymbol{H^5}\delta_{s5} \qquad \boldsymbol{H^6}:\boldsymbol{H^s} = \boldsymbol{H^6}\delta_{s6} \end{aligned} \tag{30}$$

Using Hill basis tensors, stiffness tensor $\boldsymbol{C^\Omega}$ for transversely isotropic cylindrical inclusion may be represented in form (31).

$$\begin{aligned} \boldsymbol{C^\Omega} &= (C_{22}+C_{23})\boldsymbol{H^1} + C_{12}\boldsymbol{H^2} + C_{12}\boldsymbol{H^3} + C_{11}\boldsymbol{H^4} + \\ &+ \frac{1}{2}(C_{22}-C_{23})\boldsymbol{H^5} + C_{44}\boldsymbol{H^6} \end{aligned} \tag{31}$$

From now, elements of inclusion stiffness tensor $\boldsymbol{C^\Omega}$ will be denoted as C_{ij}, whereas elements of matrix stiffness tensor, which is isotropic, will be represented as functions of Lamé constants λ and μ.

Elements of Eshelby tensor $\boldsymbol{S_{cyl}}$ for cylindrical inclusion $\boldsymbol{\Omega}$ in isotropic matrix are given by equation (14). Using Hill basis tensors, tensor $\boldsymbol{S_{cyl}}$ can be rewritten in form (32).

$$\boldsymbol{S_{cyl}} = \frac{\lambda+\mu}{2(\lambda+2\mu)}\boldsymbol{H^1} + \frac{\lambda}{2(\lambda+2\mu)}\boldsymbol{H^2} + \frac{\lambda+3\mu}{4(\lambda+2\mu)}\boldsymbol{H^5} + \frac{1}{2}\boldsymbol{H^6} \tag{32}$$

Inverse effective stiffness tensor $\bar{\boldsymbol{D}} = \bar{\boldsymbol{C}}^{-1}$ is computed similarly to the case, when particles were distributed randomly from equation (16) where Eshelby tensor $\boldsymbol{S}$ is substituted by tensor $\boldsymbol{S_{cyl}}$.
To derive it, it is necessary to represent isotropic stiffness tensor $\boldsymbol{C}$ and inverse stiffness tensor $\boldsymbol{D}$ of matrix as a linear combination of Hill basis tensor. Tensors $\boldsymbol{E^1}$ and $\boldsymbol{E^2}$ introduced in section 3 may be written now in form (33) and (34).

$$\boldsymbol{E^1} = \frac{1}{3}(2\boldsymbol{H^1} + \boldsymbol{H^2} + \boldsymbol{H^3} + \boldsymbol{H^4}) \tag{33}$$

$$\boldsymbol{E^2} = \frac{1}{3}\boldsymbol{H^1} - \frac{1}{3}\boldsymbol{H^2} - \frac{1}{3}\boldsymbol{H^3} + \frac{2}{3}\boldsymbol{H^4} + \boldsymbol{H^5} + \boldsymbol{H^6} \tag{34}$$

Whence tensors $\boldsymbol{C}$ and $\boldsymbol{D}$ may be rewritten in forms (35) and (36).

$$\boldsymbol{C} = \frac{2\lambda + 6\mu}{3}\boldsymbol{H^1} + \frac{\lambda}{3}\boldsymbol{H^2} + \frac{\lambda}{3}\boldsymbol{H^3} + \frac{\lambda}{3}\boldsymbol{H^4} + \frac{\lambda + 6\mu}{3}\boldsymbol{H^5} + 2\mu\boldsymbol{H^6} \tag{35}$$

$$\begin{aligned}\boldsymbol{D} = {} & \frac{1}{6\mu(\lambda + 2\mu)}\Big((\lambda + 6\mu)\boldsymbol{H^1} - \lambda\boldsymbol{H^2} - \lambda\boldsymbol{H^3} + 2(\lambda + 3\mu)\boldsymbol{H^4}\Big) \\ & + \frac{1}{2\mu}(\boldsymbol{H^5} + \boldsymbol{H^6})\end{aligned} \tag{36}$$

It is easy to see that fourth order symmetric unit tensor $\mathbf{1}^{(4s)}$ has form (37).

$$\mathbf{1}^{(4s)} = \boldsymbol{H^1} + \boldsymbol{H^4} + \boldsymbol{H^5} + \boldsymbol{H^6} \tag{37}$$

Therefore, one may show that if tensor $\boldsymbol{A}$ is represented as a linear combination of Hill basis tensors $\boldsymbol{A} = \sum_{k=1}^{6} h_k \boldsymbol{H^k}$ then tensor $\boldsymbol{A^{-1}}$ has form (38) [3].

$$\begin{aligned}\boldsymbol{A^{-1}} = {} & \frac{2h_4}{h_1h_4 - 2h_2h_3}\boldsymbol{H^1} - \frac{2h_2}{h_1h_4 - 2h_2h_3}\boldsymbol{H^2} - \frac{2h_3}{h_1h_4 - 2h_2h_3}\boldsymbol{H^3} \\ & + \frac{2h_1}{h_1h_4 - 2h_2h_3}\boldsymbol{H^4} + \frac{1}{h_5}\boldsymbol{H^5} + \frac{1}{h_6}\boldsymbol{H^6}\end{aligned} \tag{38}$$

Using above relations, one may show that effective tensor $\bar{\boldsymbol{D}}$ in case of transversely isotropic cylindrical inclusions embedded in isotropic matrix is given by equation (39).

$$\begin{aligned}\bar{\boldsymbol{D}} = {} & \sum_{k=1}^{6} d_k \boldsymbol{H^k} = \\ & \frac{\frac{|\Omega|}{|V|}\left(\frac{8\lambda(\lambda + 2\mu)}{3}(a_1 + a_2) + 18\mu^2 - 3\mu\lambda - \lambda^2\right) + 2(\lambda + 6\mu)}{12\mu(\lambda + 2\mu)^2}\boldsymbol{H^1} \\ & + \frac{\frac{|\Omega|}{|V|}\left(\frac{4\lambda(\lambda + 3\mu)}{3}(a_1 + a_2) + \lambda(4\lambda\mu + 8\mu^2 + 5\mu - 3\lambda\right) + 2\lambda}{12\mu(\lambda + 2\mu)}\boldsymbol{H^2} \\ & + \frac{\frac{|\Omega|}{|V|}(2\mu(2a_3 + a_4) + 2\mu(\lambda + 6\mu)) - \lambda}{6\mu(\lambda + 2\mu)}\boldsymbol{H^3} \\ & + \frac{\frac{|\Omega|}{|V|}(-\lambda\mu(2a_3 + a_4) - 2\lambda\mu) + \lambda + 3\mu}{6\mu(\lambda + 2\mu)}\boldsymbol{H^4} \\ & + \left(\frac{1}{2\mu} - \frac{|\Omega|}{|V|}\left(2a_5 - \frac{\lambda + 3\mu}{4\mu(\lambda + 2\mu)}\right)\right)\boldsymbol{H^5} + \left(\frac{1}{2\mu} - \frac{|\Omega|}{|V|}\left(2a_6 - \frac{1}{4\mu}\right)\right)\boldsymbol{H^6}\end{aligned} \tag{39}$$

where $a_k, k = 1, \ldots, 6$ are coefficients of tensor $\boldsymbol{A} = (\boldsymbol{C} - \boldsymbol{C^\Omega})^{-1}$ in representation as a linear combination of Hill basis tensors $\boldsymbol{H^k}$. Their values are given by (40).

$$
\begin{aligned}
a_1 &= \frac{3C_{11} - \lambda}{2\lambda C_{11} + (3C_{11} - \lambda)(2\mu - C_{22} - C_{23})},\\
a_2 = a_3 &= \frac{\lambda - 3C_{12}}{(2\mu - C_{22} - C_{23})(\lambda - 3C_{11}) + 2\lambda C_{11}},\\
a_4 &= \frac{2(2\lambda + 6\mu - C_{22} - C_{23})}{(2\mu - C_{22} - C_{23})(\lambda - 3C_{11}) - 2\lambda C_{11}},\\
a_5 &= \frac{2}{4\mu - C_{22} + C_{23}}, \qquad a_6 = \frac{1}{2\mu - C_{44}}
\end{aligned}
\tag{40}
$$

Comparing (39) with representation of transverse isotropic stiffness tensor (31) leads to the conclusion that derived tensor $\overline{\boldsymbol{D}}$ is not transverse isotropic. Even though it may be represented with 6 independent coefficients D_{ij} in Voigt notation for orthotropic fourth order tensors, which may be calculated from equation (41).

$$
\begin{aligned}
&\overline{D_{11}} = d_4, \qquad \overline{D_{12}} = \overline{D_{13}} = d_2, \qquad \overline{D_{21}} = \overline{D_{31}} = d_3\\
&\overline{D_{22}} = \frac{1}{2}(d_1 + 2d_2), \quad \overline{D_{23}} = \frac{1}{2}(d_1 - 2d_2), \quad \overline{D_{44}} = d_6
\end{aligned}
\tag{41}
$$

Taking into account interactions between inhomogeneities in non-dilute case, enhanced $\overline{\boldsymbol{D}}$ tensor coefficients may be computed via self-consistent method by repeating above procedure. Unlike the case with randomly distributed particles when effective stiffness tensor is still isotropic, now tensor $\boldsymbol{C}$ of matrix must be substituted with transversely isotropic tensor $\overline{\boldsymbol{D}}^{-1}$**,** what consequently complicate computations of effective stiffness tensor accounting interactions between fibres in composite.
One more major obstacle to overcome in order to compute coefficients of effective tensor $\overline{\boldsymbol{D}}$ is to find coefficients of stiffness tensor $\boldsymbol{C}$ of inclusion chain treated as transversely isotropic cylindrical fiber. Derivation of them would exceed volume of this paper, therefore it is omitted. It requires analyzing energy of magnetic interaction between inclusions in chain treated as dipoles caused by displacement of a single particle, similarly to procedure described in [7].

5. Conclusions

Methodology of computing effective stiffness tensors of magnetoelastic composites in both states, with no external magnetic field and when such field is applied was described. Treating chains formed by particles due to the magnetic interactions between them as rigid cylindrical fibers gave opportunity to derive effective elastic properties of composite in macroscale using well known homogenization methods basing on Eshelby tensor and self-consistent method. Such approach is valid only with assumptions of finite strain and linear dependence between stress and strain in chains. The second one is satisfied when particles are treated as single magnetic dipoles in state of magnetic saturation, when magnetization is constant.
Direct comparison between elastic properties in both states of work requires of course estimating stiffness tensor elements for assumed fibers. It might be an element of further research. Moreover, presented method should be also verified experimentally, especially because of simplifying assumptions, which were necessary to use standard homogenization methods valid for linear elasticity.

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