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Streszczenie: W artykule opisano nieliniowe układy wibracyjne o czystej kubicznej sile przywracania. Nie oczekuje się pełnej odpowiedzi analitycznej. Zamiast tego stosowana jest symulacja, a analiza wymiarowa jest prezentowana jako skuteczny sposób na zmniejszenie liczby zmiennych. Opisano reżimy działania izolacji drgań.

Słowa kluczowe: oscylacja nieliniowa, analiza wymiarowa, histereza, zjawiska skoku

SIMPLE APPROACH FOR PURE CUBIC NONLINEAR VIBRATING SYSTEMS

Summary: Nonlinear vibrating systems with pure cubic restoring force are described in this article. Complete analytical response is not sought. Instead, simulation is used and dimensional analysis is presented as a powerful way of reducing the number of variables. Operating regimes for vibration isolation are described.

Keywords: nonlinear oscillation, dimensional analysis, hysteresis, jump phenomena

1. Introduction

Vibrations are part of the man-machine-environment system [1]. They are characteristic of mechanical systems in which conversion of energy or motion takes place [2] and are typically considered harmful and unwanted. Especially in the environments where humans are continuously exposed to vibrating machinery, health and safety policies need to be introduced [3]. To mitigate these negative effects, various precautions are taken [4].
The generation and elimination of vibrations have been the subject of numerous research ventures [5,6,7]. It is common in engineering practice that analytical models are too complicated and simplification takes place – e.g. nonlinear systems are often linearised around the operating point [8]. Using linearisation to model complex

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systems can be accurate enough, as presented by [9, 10, 11]. However, there are applications which require nonlinear models, because linearisation is not accurate enough, or impossible whatsoever. A very common example of nonlinear systems which is often the subject of textbook analysis [12,13,14] is governed by the *Duffing* equation, in which both linear and nonlinear terms of the restoring force are present. Many of the standard methods are not applicable to *pure* cubic systems in which no linear restoring force is present. Hence, there is a need to consider such systems. Solutions to pure cubic Duffing equation systems including elliptical integrals [13] and approximate methods of harmonic balance [14] have been found. A much simpler approach is presented in this article.
We begin by a brief review of second order linear systems. A nonlinear pure cubic system is introduced via a simulation, describing its key features. Dimensional analysis is used to reduce the number of variables considerably, making it possible to obtain a complete characteristic of the system by non-dimensionalising the results of a few simulations. A simple analytical estimate is then provided, providing theoretical grounds for the observed jump phenomena.

2. Second order linear systems – a brief introduction

Most systems in mechanical engineering are of second order, whereby applied force produces acceleration – i.e. second derivative of position. Especially in the field of mechanical vibrations, equations of motion can often be derived in the classical form:

$$m\ddot{q} + \lambda\dot{q} + kq = f_{ext} \tag{1}$$

where q is a generalised coordinate, m the inertial term, λ damping, k stiffness and f_{ext} the external force.
It is useful to convert this equation into the following form with dimensionless coefficients:

$$\frac{\ddot{q}}{\omega_n^2} + \frac{2\zeta}{\omega_n}\dot{q} + q = x \tag{2}$$

where: $\omega_n = \frac{k}{m}$ natural frequency [rad.s^{-1}]

$\zeta = \frac{\lambda}{2\sqrt{km}}$ non-dimensional damping ratio

$x = \frac{f}{k}$ force input reduced to displacement [m]

When the external force f is of a periodic character with frequency ω, $f = F\ \cos\omega t$, the solution can be simplified in the complex domain. Input is assumed to take the form $x = X\,exp\,i\omega t$, and then solution exists in the form $q = Q\exp i\omega t$, with a complex amplitude Q. Then the equation of motion changes to:

$$-\left(\frac{\omega}{\omega_n}\right)^2 Q + i\,2\zeta\left(\frac{\omega}{\omega_n}\right)Q + Q = X \tag{3}$$

Amplification can be defined as $|Q/X|$ and is given by:

$$\left|\frac{Q}{X}\right| = \frac{1}{\left(\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2\right)^{0.5}} \tag{4}$$

From Eq.2 as well as from the amplification in Eq. 3, a number of important observations can be made. There are two regimes asymptotically: low frequency regime, when $\omega/\omega_n \ll 1$, and high frequency regime when $\omega/\omega_n \gg 1$. In the low

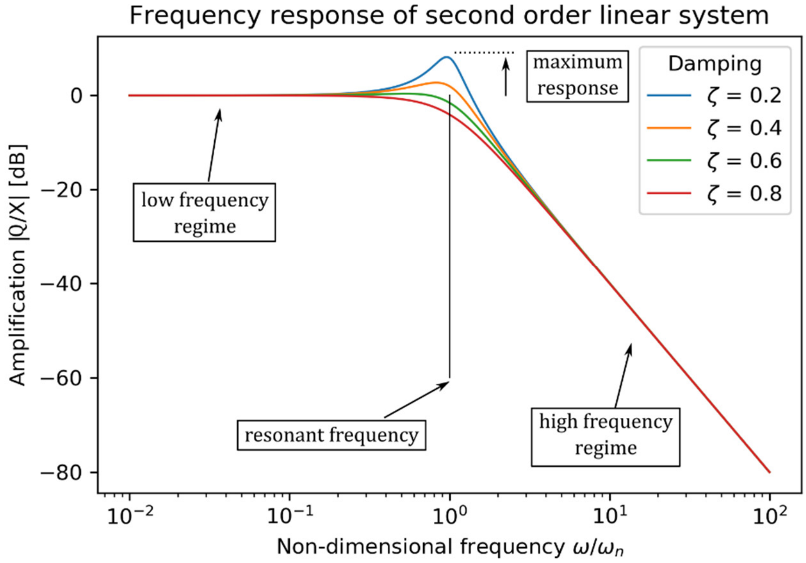

Figure 1. Frequency response of 2nd order linear system. Note the unit amplification in low-frequency regime, -40dB/decade decay in high frequency regime and decreasing maximum amplitude with increasing damping ratio.

frequency regime, the $(\omega/\omega_n)^2$ and the complex (ω/ω_n) term are vanishingly small, and we see that $Q \approx X$. This is, indeed, what one would get if static equilibrium was resolved – i.e. $q = f/k$. In the high frequency regime, the $(\omega/\omega_n)^2$ term dominates and $|Q/X| \approx 1/(\omega/\omega_n)^2$. At the resonant frequency, $\omega/\omega_n = 1$ and Q is imaginary with magnitude $|Q/X| \approx 1/(2\zeta)$.
These conclusions are proven when we plot the amplification as a function of frequency according to Eq.3 (Fig.1). They were also proven by simulations that were carried out but are not shown or described here for brevity.

3. Second order pure cubic non-linear systems

The behaviour of a system in which restoring force is proportional to the third power of displacement was observed. The governing equation was chosen to be:

$$m\ddot{q} + \lambda\dot{q} + \mu q^3 = f_{ext} \tag{5}$$

where μ $[\mathrm{N/m^3}]$ is a „stiffness" parameter of the restoring force.
The stiffness in such system is no more a constant, but rather $k = \partial f/\partial q \propto q^2$. In other words, it increases with increasing displacement and the system exhibits stiffening behaviour.
Valuable insight can be achieved by a careful inspection of Eq.4. First we consider the case of free vibration $f_{ext} = 0$. If we neglect damping for a moment (λ = 0),

the system does not dissipate energy and if initial displacement is introduced and the mass is then released, it should continue to oscillate indefinitely. Since the stiffness increases with increasing displacement, we expect that the resonant frequency to increase when the input amplitude increases.
Now external driving force is introduced. In high frequency regime, we expect inertial terms to dominate. Therefore, the μq^3 term can be neglected and the equation of motion becomes linear. Behaviour identical to linear systems is expected in high frequency regime. In low frequency regime, inertial and damping terms can be neglected, and the loading conditions are quasi-static: $\mu q^3 = f_{ext}$. Because the system exhibits stiffening behaviour, we expect the output amplitude to increase less than in proportion to the input amplitude.

3.1. Simulation results

Computer simulations were carried out to verify the predictions. Custom code was written in C++ with fourth-order Runge-Kutta integrator. Frequency sweep was used, whereby the frequency of driving force increased from $\omega = 10^{-2}$ to $\omega = 10^{2}$ and then decreased back to the initial value. The results are displayed in Fig.2. All of the asymptotical predictions were confirmed. What was not possible to predict from a simple analysis of the equation of motion is the sudden jump in the amplitude of response which occurs around resonance.

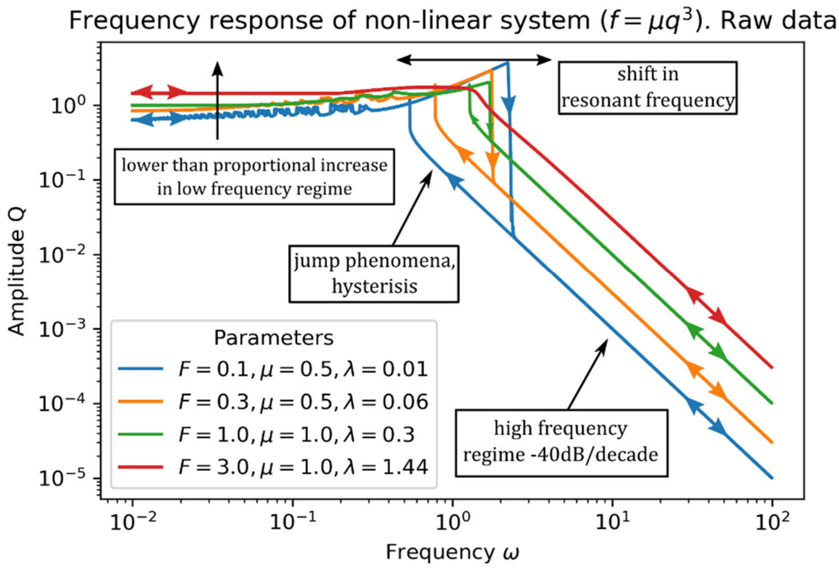

Figure 2. Frequency response of 2nd order non-linear system. Restoring force is proportional to third power of displacement. Note that with higher input amplitude, the resonant frequency is higher. In low frequency regime, the increase in amplitude is lower than in proportion to the input. System behaves linearly in high frequency regime with a -40dB/decade decay. Also note the sudden jumps which occur during the frequency sweep. Parameter $m = 1.0$ *was used in all four simulations.*

During the increasing phase of frequency sweep, the amplitude plummets by a factor of more than 10, which is quite significant. During the decreasing phase od frequency sweep, the linear regime ($-40\mathrm{dB/decade}$) is preserved longer and an opposite jump in amplitude occurs at a lower frequency.
In contrast to the plot in Fig.1, there appears to be little order in the data. However, it should be noted that the consistency that is displayed in the plot of the linear system in Fig.1 is due to a convenient choice of non-dimensional quantities that are plotted.

3.2. Dimensional analysis

Dimensional analysis can be used in the non-linear case to reduce the number of variables. There are 6 variables in the problem: mass m, damping λ, stiffness μ, driving frequency ω, amplitude of driving force F and amplitude of response Q. The number of dimensions is 3: mass, length, time. Using Buckingham's Pi Theorem, the problem can be reduced to $6 - 3 = 3$ independent dimensionless groups. We can use physical insight to form such groups. We shall proceed in the following way. First assume that there is neither damping λ, nor restoring force μq^3. Four variables in 3 dimensions remained in the problem: m, ω, F, Q. There remains only one independent dimensionless group, which has to be constant, indeed:

$$\frac{Q\omega^2 m}{F} \tag{6}$$

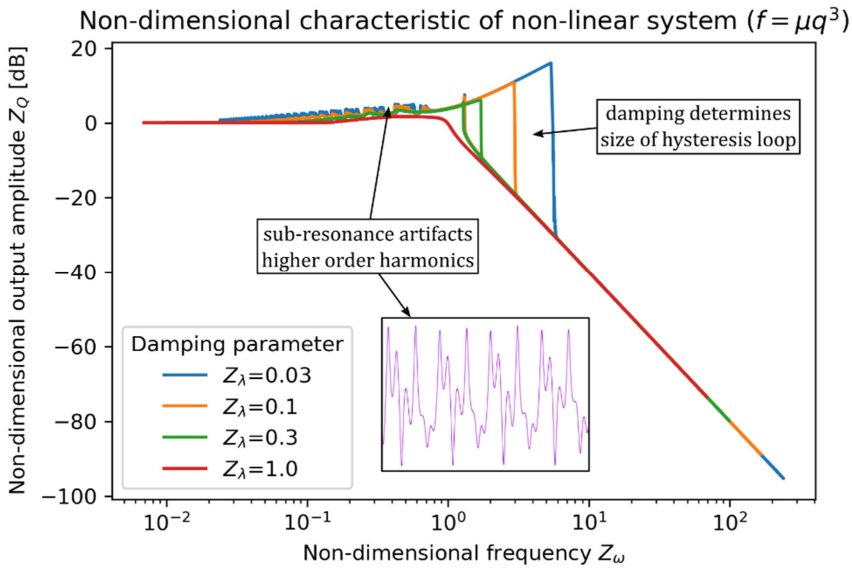

Figure 3. Non-dimensional characteristic of non-linear system. Note how all the curves collapse onto a single line in both low and high frequency regimes. Damping parameter Z_λ determines the size of the hysterisis loop. Inset shows that the irregularities in data come from higher frequency harmonics.

It turns out that this is true in high frequency regime, when stiffness and damping are negligible. Now if we introduce the restoring force, two non-dimensional groups have to be formed. Since we wish to plot amplitude against frequency, we shall keep the independent variable Q only in one of the groups, and frequency ω in the other one. The non-dimensional group characterising frequency Z_ω is therefore:

$$Z_\omega = \frac{\omega}{\sqrt[6]{\frac{F^2\mu}{m^3}}} \tag{7}$$

With restoring force included in the problem, it is useful to choose a different dimensionless group for amplitude. A convenient combination of variables is Z_Q:

$$Z_Q = \frac{Q}{\sqrt[3]{\frac{F}{\mu}}} \tag{8}$$

Now damping will be included in the problem, so one more dimensionless group characterising damping will be formed, hence Z_λ:

$$Z_\lambda = \frac{\lambda}{\sqrt[6]{F^2\mu m^3}} \tag{9}$$

These three non-dimensional groups completely characterise the problem. If we convert the quantities from Fig.2 into their respective dimensionless forms, they will collapse to a family of curves (Fig.3). This is very similar to the familiar characteristic of second order linear system such as the one in Fig.1. Note that the size of the hysterisis loop is determined by the damping parameter Z_λ. Another interesting phenomenon is the presence of little jumps in the sub-resonant region. Time data reveals that it is due to the presence of higher frequency harmonics. Slight variations in simulation parameters give differing responses, indicating that the system behaviour in this region is *chaotic* as reported in [2]. Substantial amount of analysis could be carried out on this chaotic behaviour, however, it is beyond the scope of this article.

3.3. Analytical estimate

Amplitude of the response can also be estimated analytically using a method based on harmonic balance [3]. A trick can be used, whereby the driving force is expressed as:

$$f_{ext} = F\,cos(\omega t + \phi) = F_a\,cos\,\omega t - F_b\,sin\,\omega t \tag{10}$$

The solution is sought in the form:

$$q = Q\,cos\,\omega t \tag{11}$$

It is useful to note that $cos^3\,\omega t = \frac{3}{4} cos\,\omega t + \frac{1}{4} cos\,3\omega t$. When the trial solution is substituted to Eq.4, we ignore the $3\omega t$ term, and balance of sines and cosines gives:

$$-mQ\omega^2 + \frac{3}{4}\mu Q^3 = F_a \tag{12}$$

$$\lambda\omega Q = F_b \tag{13}$$

Adding squares of Eq. 12 and 13 gives the relationship between force amplitude F and amplitude of response Q:

$$\left[\frac{3}{4}\mu Q^3 - mQ\omega^2\right]^2 + [\lambda\omega Q]^2 = F^2 \tag{14}$$

This is a polynomial of 6th order in Q, which can be rearranged, and using the non-dimensional expressions from previous analysis, it can be expressed as:

$$\frac{9}{16}Z_Q^6 - \frac{3}{2}Z_\omega^2 Z_Q^4 + (Z_\omega^4 + Z_\lambda^2 Z_\omega^2)Z_Q^2 - 1 = 0 \quad (15)$$

Note that there are no odd powers of Z_Q, therefore, its sign does not matter. As a result, the equation is cubic in Z_Q^2 and we expect a maximum of three meaningful roots. We used numerical techniques to solve the equation for different values of parameter Z_λ (Fig.4), and the predictions were confirmed. For certain values of Z_λ and Z_ω, there are three roots, and the amplitude is said to be multivalued. Considering how the amplitude has to follow the curve in Fig.4 during a frequency sweep, there has to be the jump in amplitude, which was observed in simulation results, indeed. It has been

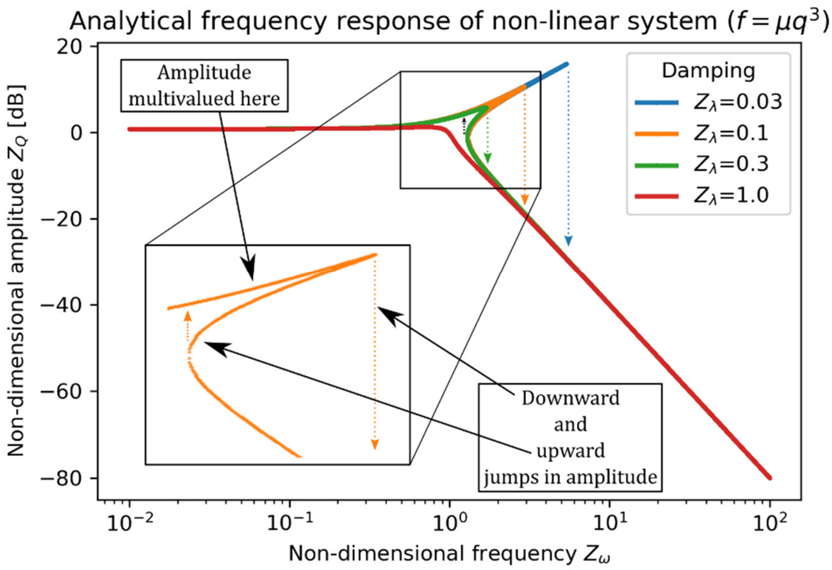

Figure 4. Analytical predictions of system response. Note in the inset that the amplitude is multivalued around resonance, but the middle value is unstable [2] and a jump occurs.

estimated using numerical methods that the function ceases to be multivalued at the values of damping parameter above $Z_\lambda \geq 0.7$.

4. Discussion

We will use the background presented in the previous sections to draw some conclusions for reduction of vibration in mechanical systems. In section 2, it was shown that amplification of input oscillations is lowest in the high frequency range. If the frequency of input is set, the system can nevertheless be designed to operate in the high frequency range. This can be done by manipulating the natural frequency of the system $\omega_n = \sqrt{k/m}$. There are two ways of doing so: increase mass m, or reduce stiffness k. Such a modification will shift the operating point to the right on the non-dimensional plot in Fig.1. It should be noted that lowering stiffness k has an effect only until resonance occurs. It was shown that once high frequency regime is reached, stiffness has no effect on the response and one would need to increase mass to further reduce the output amplitude.

It has been noted that the system with non-linear restoring force behaves linearly in the high frequency range, whereby the same $-40\mathrm{dB/decade}$ decay occurs. Therefore, the operating point can be chosen to lie within the high frequency regime. Analogous modifications to the mass m and stiffness μ can be made in order to operate in the high frequency regime. There is another useful characteristic of the non-linear system that could be exploited in design. The hysteresis jump that occurs around resonant frequency can be used as a means of locking the amplitude in the linear $-40\mathrm{dB/decade}$ regime. When frequency is increased from low frequency regime, amplitude increases up to resonance, and once the downward jump occurs, subsequent decrease of frequency will not trigger resonance again. It should be emphasised that the size of the hysteresis loop is determined by the damping parameter Z_{λ}, and there are values of this parameter for which no hysteresis loop is observed.

Since the nonlinear system exhibits stiffening behaviour, the amplitude of oscillation will be lowered in comparison to an equivalent linear system. Hence, nonlinearity can be introduced into a system on purpose to modify its vibration characteristic.

There are two distinct novel concepts of devices known to the authors at the present time, which would exploit the high frequency regime and non-linear behaviour, respectively. However, they are beyond the scope of this article and they shall be presented in the close future.

5. Conclusion

In this article, key features of second order linear and non-linear systems were presented. Attention was given specifically to the properties which could be exploited in vibration-oriented design. A number of important conclusions can be made.

(i) Contrary to the usual treatment when the Duffing equation is analysed with both a linear restoring force kq and a non-linear term μq^3, we analysed a system in which the linear part of restoring force is absent.

(ii) We presented dimensional analysis as a rather powerful tool to analyse complex systems. Using dimensional analysis, the nonlinear vibration problem was reduced from 6 variables to 3 independent non-dimensional groups, enabling plotting with one parameter in 2D. A few simulations can produce a dimensionless plot which will predict the behaviour of *any* pure cubic nonlinear system.

(iii) When the system operates in the high frequency regime, restoring force and damping become negligible, and any second order system with a constant inertia term can by reduced to one dimensionless group which has to be constant. It is desirable to operate in this region, for the output amplitude is lowest.

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